Fermion propagation through spacetime

The electron propagator is given by

$$i S_{\alpha\beta}(x) \equiv \langle 0| T[\psi_\alpha(x)\bar{\psi}_\beta(0)]|0\rangle$$

As we will see, have to define

$$T[\psi(x)\bar{\psi}(0)] = \theta(x^0)\psi(x)\bar{\psi}(0) - \theta(-x^0)\bar{\psi}(0)\psi(x)$$

sign due to anti-commuting nature of ψ

Using

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\left(\frac{E_p}{m}\right)^{\frac{1}{2}}} \sum_s \left[b(p,s)u(p,s)e^{-ipx} + d^\dagger(p,s)v(p,s)e^{ip\cdot x}\right],$$

we get for $x^0 > 0$

$$i S(x) = \langle 0|\psi(x)\bar{\psi}(0)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\left(\frac{E_p}{m}\right)^{\frac{1}{2}}} \int \frac{d^3p'}{(2\pi)^{\frac{3}{2}}\left(\frac{E_{p'}}{m}\right)^{\frac{1}{2}}} \Bigg[\langle 0|\sum_{s,s'} u(p,s)\bar{u}(p',s')e^{-ip\cdot x} \underbrace{bb^\dagger|0\rangle}_{\substack{=\{b,b^\dagger\}|0\rangle \\ = \delta^{(3)}(\vec{p}-\vec{p}')\delta_{ss'}}}$$

$$+ \langle 0|\sum_{s,s'} u(p,s)\bar{v}(p',s')e^{-ip\cdot x} \underbrace{bd|0\rangle}_{=0}$$

$$+ \underbrace{\langle 0 | d^\dagger}_{= (d|0\rangle)^\dagger = 0} \sum_{s,s'} v(p,s)\bar{u}(p',s') e^{ip\cdot x} b^\dagger |0\rangle$$

$$+ \langle 0 | \sum_{s,s'} v(p,s)\bar{v}(p',s') e^{ip\cdot x} d^\dagger \underbrace{d\,|0\rangle}_{=0}$$

$$= \int \frac{d^3p}{(2\pi)^3 (E_p/m)} \sum_s u(p,s)\bar{u}(p,s) e^{-ip\cdot x}$$

$$= \int \frac{d^3p}{(2\pi)^3 \, E_p/m} \frac{\not{p}+m}{2m} e^{-ip\cdot x} \qquad (1)$$

For $x^0 < 0$, we have to compute

$$iS_{\alpha\beta}(x) = -\langle 0 | \bar{\psi}_\beta(0)\, \psi_\alpha(x) | 0\rangle$$

$$= -\int \frac{d^3p}{(2\pi)^3 \, E_p/m} \sum_s \bar{v}_\beta(p,s) v_\alpha(p,s) e^{-ip\cdot x}$$

$$= -\int \frac{d^3p}{(2\pi)^3 \, E_p/m} \left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta} e^{-ip\cdot x} \qquad (2)$$

Putting (1) and (2) together, we get

$$iS(x) = \int \frac{d^3p}{(2\pi)^3 \, E_p/m} \left[\theta(x^0) \frac{\not{p}+m}{2m} e^{-ip\cdot x} - \theta(-x^0) \frac{\not{p}-m}{2m} e^{ip\cdot x}\right]$$

This can be written more elegantly as

$$i S(x) = i \int \frac{d^4 p}{(2\pi)^4} e^{-ip\cdot x} \frac{\not{p}+m}{p^2-m^2+i\varepsilon}$$

$$= \int \frac{d^4 p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{\not{p}-m+i\varepsilon}$$

To see this, note that in the complex p^0 plane the integrand has poles at

$$p^0 = \pm\sqrt{\vec{p}^2+m^2-i\varepsilon} \simeq \pm(E_p - i\varepsilon)$$

For $x^0 > 0$, the factor $e^{-ip^0 x^0}$ tells us to close the contour in the lower half-plane

$\longrightarrow$ going around the pole $+(E_p - i\varepsilon)$ clockwise, we get

$$i S(x) = (-i) i \int \frac{d^3 p}{(2\pi)^3} e^{-ip\cdot x} \frac{\not{p}+m}{2E_p}$$

For $x^0 < 0$, we have to close the contour in the upper half-plane

$\longrightarrow$ have to go around the pole $-(E_p - i\varepsilon)$ anticlockwise

$$\rightarrow \quad iS(x) = i^2 \int \frac{d^3p}{(2\pi)^3} e^{+iE_p x^0 + i\vec{p}\cdot\vec{x}}$$

$$\times \frac{1}{-2E_p}\left(-E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} + m\right)$$

flipping $\vec{p}$ we get

$$iS(x) = -\int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \frac{1}{2E_p}\left(E_p\gamma^0 - \vec{p}\cdot\vec{\gamma} - m\right)$$

$$= -\int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \frac{1}{2E_p}(\not{p} - m)$$

To summarize, we see that the momentum space fermion propagator has the elegant form

$$iS(p) = \frac{i}{\not{p} - m + i\varepsilon}$$

$\rightarrow$ $S(p)$ is the inverse of the Dirac operator, just as the scalar boson propagator

$$D(k) = \frac{1}{k^2 - m^2 + i\varepsilon}$$ is the inverse

of the Klein-Gordon operator $k^2 - m^2$

§ 2.4 Grassmann Integrals and Feynman Diagrams for Fermions

Vacuum energy:

By definition, vacuum fluctuations occur when there are no sources to produce particles

$\longrightarrow$ for free scalar field theory we get

$$Z = \int \mathcal{D}\varphi \, e^{i\int d^4x \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2]}$$

$$= C\left(\frac{1}{\det[\partial^2 + m^2]}\right)^{\frac{1}{2}} = C e^{-\frac{1}{2}\mathrm{Tr}\log(\partial^2 + m^2)} \qquad (1)$$

where we used the identity

$$\det M = e^{\mathrm{Tr}\log M} \qquad (2)$$

(exercise)

Recall that $Z = \langle 0 | e^{-iHT} | 0 \rangle$ (with $T \to \infty$)

$$= e^{-iET}$$

energy of vacuum

Let us now evaluate the trace in (1)

this can be done by noting

$$\mathrm{Tr}\, G = \int d^4x \, \langle x | G | x \rangle$$

$$= \int d^4x \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \langle x | k \rangle \langle k | G | q \rangle \langle q | x \rangle$$

$\rightarrow$ we get:

$$ET = \frac{1}{2} VT \int \frac{d^4k}{(2\pi)^4} \log(k^2 - m^2 + i\varepsilon) + A \quad (3)$$

where A is an infinite constant corresponding to the multiplicative factor C in (1)

Let us define m' by writing

$$A = -\frac{1}{2} VT \int \frac{d^4k}{(2\pi)^4} \log(k^2 - m'^2 + i\varepsilon)$$

$\rightarrow$ (3) gives difference of vacuum energies of particles with mass m and m'

$$\rightarrow \frac{E}{V} = -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \log\left[\frac{k^2 - m^2 + i\varepsilon}{k^2 - m'^2 + i\varepsilon}\right]$$

$$= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi} \log\left[\frac{\omega^2-\omega_k^2+i\varepsilon}{\omega^2-\omega_k'^2+i\varepsilon}\right]$$

where $\omega_k' = +\sqrt{\vec{k}^2+m^2}$

Integrating by parts, we get

$$\int \frac{d\omega}{2\pi} \log\left[\frac{\omega^2-\omega_k^2+i\varepsilon}{\omega^2-\omega_k'^2+i\varepsilon}\right]$$

$$= -2\int \frac{d\omega}{2\pi}\,\omega\left[\frac{\omega}{\omega^2-\omega_k^2+i\varepsilon} - \frac{\omega}{\omega^2-\omega_k'^2+i\varepsilon}\right]$$

$$= -i2\omega_k^2\left(\frac{1}{-2\omega_k}\right) - (\omega_k \to \omega_k')$$

$$= +i(\omega_k - \omega_k')$$

Restoring $\hbar$, we then arrive at

$$\frac{E}{V} = \int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{2}\hbar\omega_k - \frac{1}{2}\hbar\omega_k'\right) \qquad (4)$$

$\longrightarrow$ this matches with the result previously obtained using the operator formalism (see §1.1)

A peculiar sign for fermions

Recall that for fermions, the operator formalism gave a different sign (see § 2.3)

Question: How can we obtain this from the path integral formalism?

Recall that the origin of the sign lies in Gaussian integration result

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} = \sqrt{2\pi}\, e^{-\frac{1}{2}\log a}$$

→ need to find new type of integral giving $e^{+\frac{1}{2}\log a}$

Grassmann variables

We postulate, following Grassmann (19th century), the existence of a new kind of number, called Grassmann or anticommuting number:

$$\eta \xi = - \xi \eta$$

$$\rightarrow \quad \eta^2 = 0$$

functions of η are of general form:

$$f(\eta) = a + b\eta$$

postulate translation inv. of integral:

$$\underbrace{\int_{-\infty}^{\infty} d\eta\, f(\eta+\xi)} = \underbrace{\int_{-\infty}^{\infty} d\eta\, f(\eta)}$$

$$= \int_{-\infty}^{\infty} d\eta \left(a + b(\eta+\xi)\right) = \int_{-\infty}^{\infty} d\eta \left(a + b\eta\right)$$

$$\rightarrow \int_{-\infty}^{\infty} d\eta\, b\xi = 0$$ for any Grassmann number ξ

thus we conclude: $\int d\eta\, b = 0 \rightarrow \int d\eta\, 1 = 0$

ordinary number

Since $\int d\eta\, \eta$ behaves like ordinary number, we define $\int d\eta\, \eta = 1$